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Closed conformal Killing-Yano tensor and uniqueness of generalized Kerr-NUT-de Sitter spacetime

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I. Introduction to KND spacetime

II. Geodesic integrability

III. Uniqueness of KND spacetime

IV. Summary and Discussion

We focus on rotating black hole solutions with spherical horizon to vacuum Einstein Eq..

$$R_{ab} = \lambda g_{ab}$$

In 4 dimension

Schwarzschild (1916), Kerr (1963), Carter (1968), Plebanski (1975)

In higher dimensions

Tanghelini (1916), Myers-Perry (1986), Hawking-Hunter-Taylor-Robinson (1998), Gibbons-Lu-Page-Pope (2004), Chen-Lu-Pope (2006)

The most general known solution ↑

= Kerr-NUT-de Sitter (KND) spacetime

- D=2n or 2n+1 KND metric ($\varepsilon = 0$ or 1)

$$ds^2 = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k\right]^2 + \varepsilon \frac{c}{A^{(n)}} \left[\sum_{k=0}^{n} A^{(k)} d\psi_k\right]^2 ,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2) , \quad \boxed{X_\mu = X_\mu(x_\mu) ,}$$

← **We can determine these functions with Einstein Eq.**

$$A_\mu^{(k)} = \sum_{\substack{1\leq\nu_1<\nu_2<\cdots<\nu_k\leq n \\ \nu_i\neq\mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1\leq\nu_1<\nu_2<\cdots<\nu_k\leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 ,$$

$$A_\mu^{(0)} = A^{(0)} = 1 , \quad c = \text{const.} .$$

when $X_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k} + b_\mu x_\mu$ in 2n dimension

$X_\mu = \sum_{k=1}^{n} c_k x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}$ in 2n+1 dimension ,

This metric satisfies Einstein Eq. $R_{ab} = -(D-1)c_n\, g_{ab}$.

- Various field equations on KND background are integrable.

  - Geodesic equation

  - Klein-Gordon equation

  - Dirac equation

  - Equation for gravitational perturbation !

  - Maxwell equation ?

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Killing tensor and conserved quantity

There are two ways to generalize Killing vector and conformal Killing vector.

<u>Killing Eq.</u>

For a rank-n symmetric tensor $K_{a_1 \dots a_n}$

$$\nabla_{(b} K_{a_1 \dots a_n)} = 0$$

$$C_K = K_{a_1 \dots a_n} p^{a_1} \dots p^{a_n}$$

(K : Killing tensor , p : geodesic tangent)

$\Rightarrow$ C_K is constant along geodesic

$$\because \; p^a \nabla_a C_K = 0$$

Killing-Yano tensor

KY Eq.

For a rank-n anti-symmetric tensor $h_{a_1 \ldots a_n}$

$$\nabla_{(a} h_{b) c_1 \ldots c_{n-1}} = 0$$

- $K_{ab} = f_{a c_1 \cdots c_n} f_b{}^{c_1 \cdots c_n}$: rank-2 Killing tensor

Conformal Killing-Yano (CKY) tensor

CKY Eq.

Tachibana and Kashiwada (1968)

For a rank-n anti-symmetric tensor $h_{a_1 \dots a_n}$

$$\nabla_{(a} h_{b)c_1 \dots c_{n-1}} = g_{ab} \xi_{c_1 \dots c_{n-1}} + \sum_{i=1}^{n-1} (-1)^i g_{c_i (a} \xi_{b) c_1 \dots \hat{c}_i \dots c_{n-1}}$$

where $\xi_{c_1 \dots c_{n-1}} = \frac{1}{D - n + 1} \nabla^a h_{a c_1 \dots c_{n-1}}$

- closed CKY tensor h $\xleftrightarrow[f = *h]{\text{Hodge dual}}$ Killing-Yano tensor f

  $dh = 0$ i.e. $\nabla_{(a} f_{b) c_1 \dots c_n} = 0$

- closed CKY ∧ closed CKY = closed CKY

Geodesic motion in 4D Kerr spacetime

There exist three trivial constants and one more non-trivial constant of geodesic motion in 4D Kerr spacetime.

Carter (1968)

# dimension	# Killing vector	# Killing tensor
4	2	2

time translation

$$\xi^a = (\frac{\partial}{\partial t})^a$$

axial symmetry

$$\eta^a = (\frac{\partial}{\partial \phi})^a$$

metric

$$g_{ab}$$

non-trivial Killing tensor

$$K_{ab}$$

After that, it was shown that the Killing tensor can be written as the square of a rank-2 Killing-Yano tensor.

Penrose, Floyd (1973)

$$^{\exists}f \quad \text{s.t. } K_{ab} = f_a{}^c f_{bc}, \quad \underline{f_{ba} = -f_{ab}, \ \nabla_{(a}f_{b)c} = 0}$$

↑ rank-2 KY condition

Moreover, two Killing vectors are generated from the Killing-Yano tensor.

Hughston, Sommers (1987)

$$\xi^a \equiv \nabla_b (*f)^{ba} = (\partial_t)^a$$

$$\eta^a \equiv K^a{}_b \xi^b = (\partial_\phi)^a$$

It seems that Killing-Yano tensor is more fundamental.

Geodesic motion in higher dimensions

Since KND spacetime has time translation and axial symmetries, there exist n or n+1 Killing vectors in 2n or 2n+1 dimension.

And there exists a rank-2 closed conformal Killing-Yano tensor. Then n-1 Killing tensors besides metric are generated from it.

Frolov et al (2006)

# dimension	# Killing vector	# Killing tensor
2n	n	n
2n+1	n+1	n

Theorem 1

We assume that D-dimensional spacetime (M, g) admits a single rank-2 non-degenerate closed CKY tensor which satisfies

$$*\mathbf{1}\ \mathcal{L}_\xi g = 0\ ,\quad *\mathbf{2}\ \mathcal{L}_\xi h = 0\ .$$

Then the geodesic equation on (M, g) is integrable .

∗ We actually don't have to assume ***1** and ***2** because

$$\mathrm{d}h = 0 \quad \Rightarrow \quad *\mathbf{1}\ \mathcal{L}_\xi g = 0\ ,\quad *\mathbf{2}\ \mathcal{L}_\xi h = 0\ .$$

Frolov, Krtous, Kubiznak and Page (2007)

Outline of proof

rank-2 closed CKY

h

▸ rank-2j closed CKY ▸ rank-(D-2j) KY

$$h^{(j)} = h \wedge \ldots \wedge h \qquad f^{(j)} = *h^{(j)}$$

▸ rank-2 Killing tensor ▸ conserved quantity

$$K^{(j)}_{ab} = f^{(j)}_{a\cdots} f_b^{(j)\cdots} \qquad C_j = K^{(j)}_{ab} p^a p^b$$

---------------------------------- ▼ ----------------------------------

*1 ▸ Killing vector *2 ▸ Killing vector ▸ conserved quantity

$$\xi_a = \nabla^b h_{ba} \qquad \eta^{(j)}_a = K^{(j)}_{ab} \xi^b \qquad F_j = \eta^{(j)}_a p^a$$

It can be checked that

$$\{C_i, C_j\} = 0 \ ,$$ Krtous, Kubiznak, Page and Frolov (2006)

$$\{F_i, F_j\} = 0 \ , \ \{C_i, F_j\} = 0 \ .$$ Houri, Oota and Yasui (2007)

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Question :

Do there exist any other spacetimes with a rank-2 non-degenerate closed CKY tensor except for KND spacetime?

Theorem 2

Houri, Oota and Yasui (2007)

We assume that D-dimensional spacetime (M, g) admits a single rank-2 non-degenerate closed CKY tensor. Then (M, g) is the only ''KND spacetime''.

- D=2n or 2n+1 "KND metric" ($\varepsilon = 0$ or 1)

$$ds^2 = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon \frac{c}{A^{(n)}} \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2 ,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (x_\mu^2 - x_\nu^2) , \quad \boxed{X_\mu = X_\mu(x_\mu) ,}$$

We can't determine these functions without Einstein Eq.

$$A_\mu^{(k)} = \sum_{\substack{1 \le \nu_1 < \nu_2 < \cdots < \nu_k \le n \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1 \le \nu_1 < \nu_2 < \cdots < \nu_k \le n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 ,$$

$$A_\mu^{(0)} = A^{(0)} = 1 , \quad c = \text{const.} .$$

- rank-2 closed CKY

$$h = \sum_{\mu}^{n} x_\mu e^\mu \wedge e^{\mu+n}$$

$\{e^a\}$: orthonormal basis

Eigenvalues are functions : non-degenerate

In general, we may have arbitrary values as eigenvalues of a rank-2 closed CKY tensor.

Theorem 3

Houri, Oota and Yasui (2008)

We assume that D-dimensional spacetime (M, g) admits a single rank-2 closed CKY tensor. Then (M, g) is the only ''generalized KND spacetime''.

It is convenient to write eigenvalues of a rank-2 closed CKY by introducing $Q^a{}_b = -h^a{}_c h^c{}_b$.

$$V^{-1}(Q^a{}_b)V = \{\underbrace{x_1^2, x_1^2, \dots, x_n^2, x_n^2}_{2n}, \underbrace{\xi_1^2, \dots, \xi_1^2}_{2m_1}, \dots, \underbrace{\xi_N^2, \dots, \xi_N^2}_{2m_N}, \underbrace{0, \dots, 0}_{K}\}$$

Then D-dimensional ''generalized KND metric'' is

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} \theta_k\right]^2 + \sum_{j=1}^{N} \prod_{\mu=1}^{n} \boxed{(x_\mu^2 - \xi_j^2) g^{(j)}} + \boxed{(\prod_\mu x_\mu^2) g^{(0)}}$$

where $g^{(0)}$ is arbitrary K-dim metric and $g^{(j)}$ is $2m_j$-dim Kahler metric with the Kahler form $\omega^{(j)}$.

$$P_\mu = \frac{\boxed{X_\mu(x_\mu)}}{x_\mu^K \prod_{j=1}^{N} (x_\mu^2 - \xi_j^2)^{m_j} \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2)}, \quad A_\mu^{(k)} = \sum_{\nu_i \neq \mu} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2$$

$$d\theta_k + 2\sum_{j=1}^{N} (-1)^{n-k} \xi_j^{2n-2k-1} \omega^{(j)} = 0$$

We can't determine them any more without Einstein Eq.

- D-dimensional generalized KND metric

$$g=\sum_{\mu=1}^{n}\frac{dx_\mu^2}{P_\mu}+\sum_{\mu=1}^{n}P_\mu\left[\sum_{k=0}^{n-1}A_\mu^{(k)}\theta_k\right]^2+\sum_{j=1}^{N}\prod_{\mu=1}^{n}(x_\mu^2-\xi_j^2)g^{(j)}+(\prod_\mu x_\mu^2)g^{(0)}$$

where

$$P_\mu=\frac{X_\mu(x_\mu)}{x_\mu^K\prod_{j=1}^N(x_\mu^2-\xi_j^2)^{m_j}\prod_{\substack{\nu=1\\ \nu\neq\mu}}^n(x_\mu^2-x_\nu^2)}\,,\quad A_\mu^{(k)}=\sum_{\nu_i\neq\mu}x_{\nu_1}^2x_{\nu_2}^2\dots x_{\nu_k}^2$$

$$d\theta_k+2\sum_{j=1}^{N}(-1)^{n-k}\xi_j^{2n-2k-1}\omega^{(j)}=0$$

When $g^{(0)}$ is K-dim Einstein metric, $g^{(j)}$ is $2m_j$-dim Einstein-Kahler metric with the Kahler form $\omega^{(j)}$ and

$$X_\mu=x_\mu\int dx_\mu\ \chi(x_\mu)x_\mu^{K-2}\prod_{i=1}^{N}(x_\mu^2-\xi_i^2)^{m_i}\ +\ d_\mu x_\mu$$

where

$$\chi(x_\mu)=\sum_{i=0}^{n}\alpha_i x^{2i}\,,\quad \alpha_0=(-1)^{n-1}\lambda^{(0)}\qquad \left(\ \lambda^{(j)}=(-1)^{n-1}\chi(\xi_j^2)\ \right)$$

This metric satisfies Einstein Eq. $R_{ab}=-(D-1)\alpha_n\, g_{ab}$.

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Summary

The geodesic equation on a spacetime with a rank-2 non-degenerate closed conformal Killing-Yano tensor is integrable.

However, such a spacetime is the only ''KND spacetime''.

In general, without non-degenerate we have ''generalized KND spacetime''.

＊non-degenerate means all the eigenvalues of CKY tensor are functions.

Discussion

Test Maxwell field on KND background

Stability of KND spacetime

Generalized CKY tensor