Black hole instabilities and local Penrose inequalities

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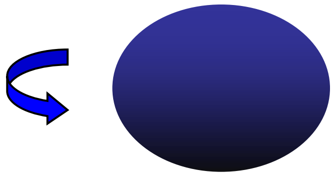
1.Introduction

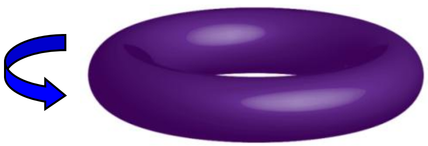
Motivation of stability analysis of higher dimensional BHs

black holes in higher dimensions

In higher dimensions, there are many kinds of BHs.



Myers-Perry BH S^{d-2}



black ring $S^{d-3} \times S^1$

What is the final state of gravitational collapse in higher dimensions? → stability analysis of higher dimensional BHs

What kinds of BHs are generated in LHC？

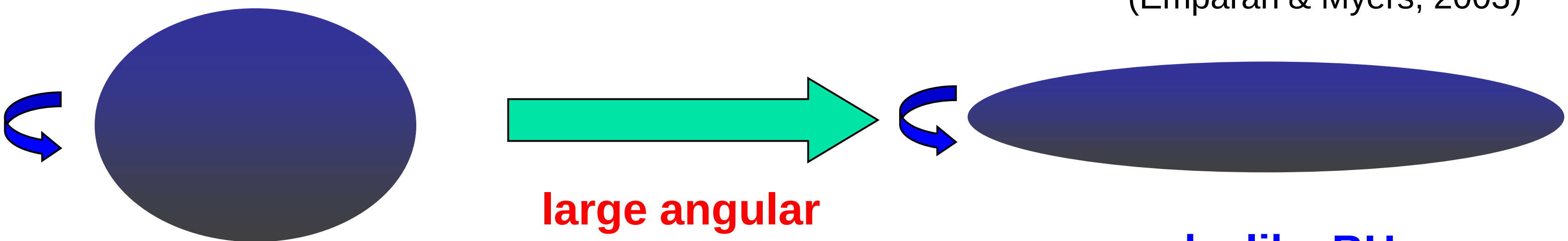
AdS/CFT correspondence

Instabilities of AdS BHs correspond to phase transitions in dual theories.

→ The stability analysis is important to understand phase structure of dual theories.

Recent progress of stability analysis (Myers-Perry BHs)

(Emparan & Myers, 2003)



large angular momentum

pancake like BH

≒ black brane

Myers-Perry BH

Gregory-Laflamme instability

This ultra-spinning instability was confirmed by numerical calculations.

Dias, Figueras, Monteiro, Reall & Santos, 10,
Dias, Figueras, Monteiro, Santos & Emparan, 09,
Yoshino and Shibata, 10

The equations governing linearized perturbations are typically very complicated.
It would be nice if there were a simpler method of demonstrating black hole instabilities.

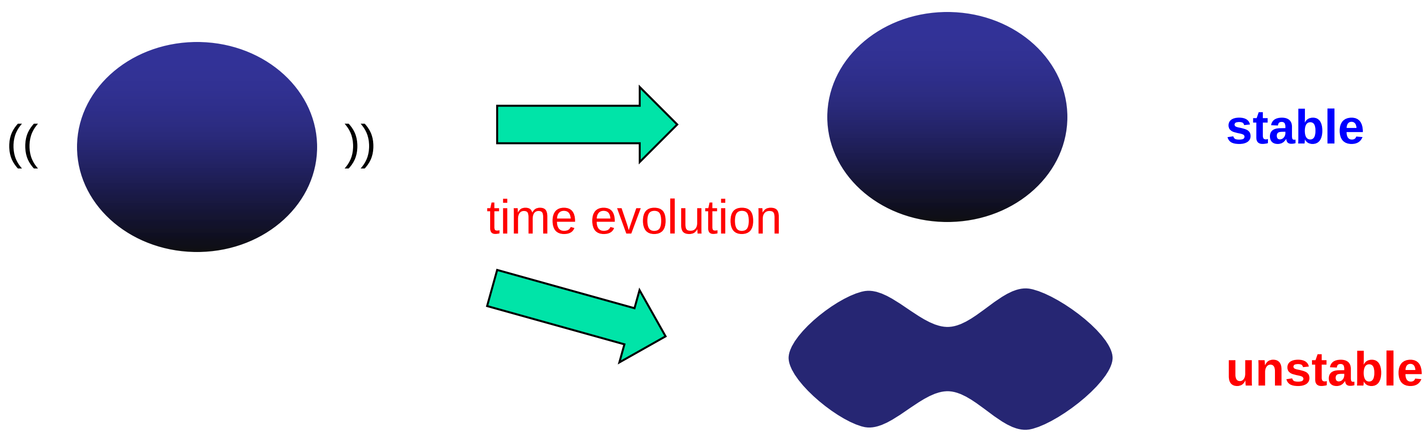
Local Penrose inequalities

2.Black hole instabilities and local Penrose inequality

What is the black hole instability?

After we give a perturbation on a black hole, the black hole evolves dynamically.

If the black hole is stable, it comes back to the original solution.
If the black hole is unstable, it goes to another solution.



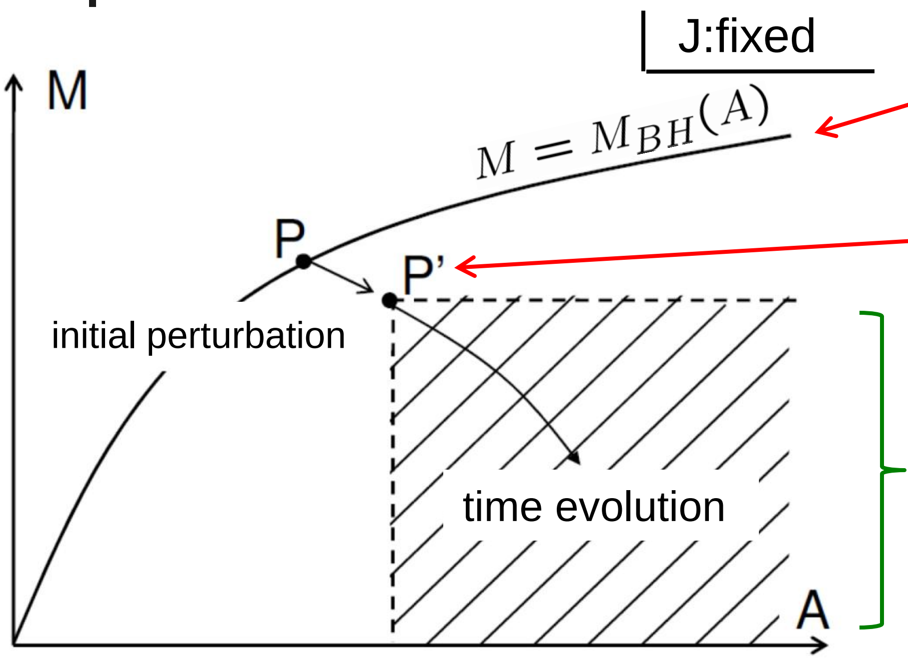
Dynamical black holes have following properties.

(1) Area of the event horizon increases.(Hawking's area theorem) $dA/dt > 0$

(2) Mass decreases. (Energy loss by gravitational radiation) $dM/dt < 0$

Bondi et al,62, Hollands&ishibashi,03, Tanabe et al,11, Godazgar&Reall,12

Our Idea



relation of mass M and area of horizon A of a background spacetime

Assume that a black hole at a point P moves to P' by an **axisymmetric** initial perturbation.

$$dA/dt > 0 \qquad dM/dt < 0$$

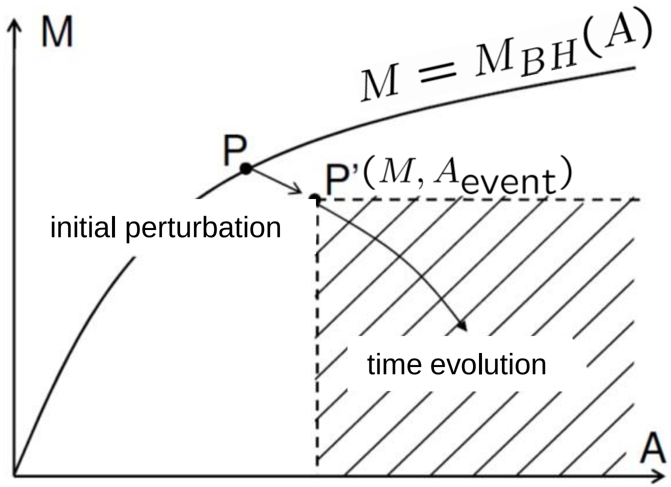
The spacetime cannot come back to the sequence of original solutions.

➡ **unstable**

Using this idea, we can show instability only from initial data of black hole perturbations.

Inequality for stable BHs

If the BH is stable, P’ must be above the curve for any axisymmetric initial perturbation.



Namely, stable balck holes satisfy a inequality:

$$M > M_{BH}(A_{\text{event}})$$

From **Area(event) >= Area(apparent)**, we have

$$\underline{M} > M_{BH}(A_{\text{event}}) > \underline{M_{BH}(A_{\text{apparent}})}$$

This inequality can be evaluated only from initial data.

local Penrose inequality

inequality for stable BHs: $M > M_{BH}(A)$

We expand this inequality perturbatively as

The number of dots represents order of perturbation.

$$M = M_0 + \dot{M} + \frac{1}{2}\ddot{M} + \cdots \qquad A = A_0 + \dot{A} + \frac{1}{2}\ddot{A} + \cdots$$

$$\underline{(\dot{M} - T\dot{A}/4)} + \frac{1}{2}\left(\ddot{M} - T\ddot{A}/4 - \frac{1}{T c_J}\dot{M}^2\right) > 0$$

= 0 (1st law Sudarsky&Wald, 92, Hawking, 73)

$$c_J = \left(\frac{\partial M}{\partial T}\right)_J$$

$$\boxed{Q \equiv \ddot{M} - T\ddot{A}/4 - \frac{1}{T c_J}\dot{M}^2 > 0}$$

We call this inequality as local Penrose inequality.

Stable black holes satisfy the local Penrose inequality.

Violation of local Penrose inequality is a sufficient condition for instability.

It was shown that violation of the inequality is also the necessary condition for instability (Hollands&Wald,12).

Strategy to find instabilities

1.We construct initial data of black hole perturbation(Up to 2nd order).

2.From the initial data, we read off the deviation of mass and area of apparent horizon.

3.Substituing them into Q, we check the sign of Q.

$$Q \equiv \ddot{M} - T\ddot{A}/4 - \frac{1}{T\,c_J}\dot{M}^2$$

4. If Q<0, we can show the instability of BHs.

Construction of initial data

Set of initial data of background solution $(\bar{h}_{ab}, \bar{K}_{ab})$

They satisfy the constraint equations:

$$\boxed{\begin{aligned} &\bar{R} + \bar{K}^2 - \bar{K}_{ab}\bar{K}^{ab} = 0 \\ &\bar{\nabla}_b \bar{K}^b{}_a - \bar{\nabla}_a \bar{K} = 0 \end{aligned}}$$

We focus on special set of initial perturbations.

$$\begin{cases} h_{ab} = \Psi^{4/(d-3)} \bar{h}_{ab}, \\ K_{ab} = \Psi^{-2} \bar{K}_{ab} \end{cases}$$

constraint eqs

$$\bar{\nabla}^2 \Psi - \frac{(d-3)}{4(d-2)} \bar{R} \left(\Psi - \Psi^{-3-4/(d-3)} \right) = 0,$$

$$\Psi = 1 + \dot{\Psi} + \frac{1}{2}\ddot{\Psi}$$

$$\boxed{\bar{\nabla}^2 \dot{\Psi} - \bar{R}\dot{\Psi} = 0,}$$

1st order eq

$$\boxed{\bar{\nabla}^2 \ddot{\Psi} - \bar{R}\ddot{\Psi} = -\left(\frac{3d-5}{d-3} \right) \bar{R} \dot{\Psi}^2}$$

2nd order eq

3.Gregory-Laflamme instability

Gregory-Laflamme instability

We check the Gregory-Laflamme instability using our method.

black string solution in isotropic coordinates

$$ds^2 = -\frac{(1 - r_0/r)^2}{(1 + r_0/r)^2} dt^2 + \left(1 + \frac{r_0}{r}\right)^4 \left(dr^2 + r^2 d\Omega_2^2\right) + dx^2 \,,$$

metric on t=0 surface: $ds^2 = \left(1 + \frac{r_0}{r}\right)^4 \left(dr^2 + r^2 d\Omega_2^2\right) + dx^2 \,,$

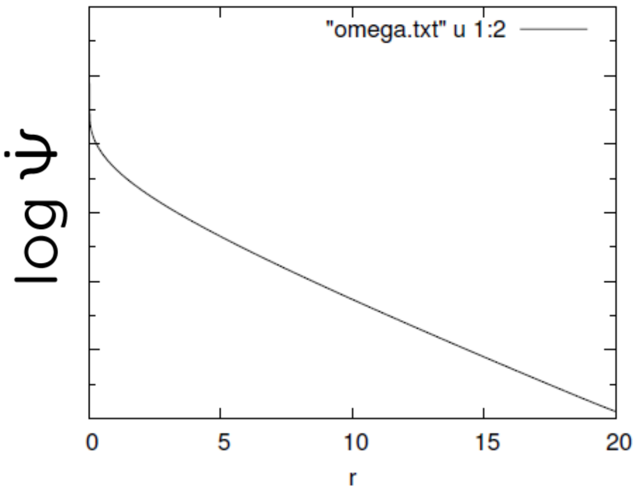
1st order eq $\bar{\nabla}^2 \dot{\Psi} = 0$

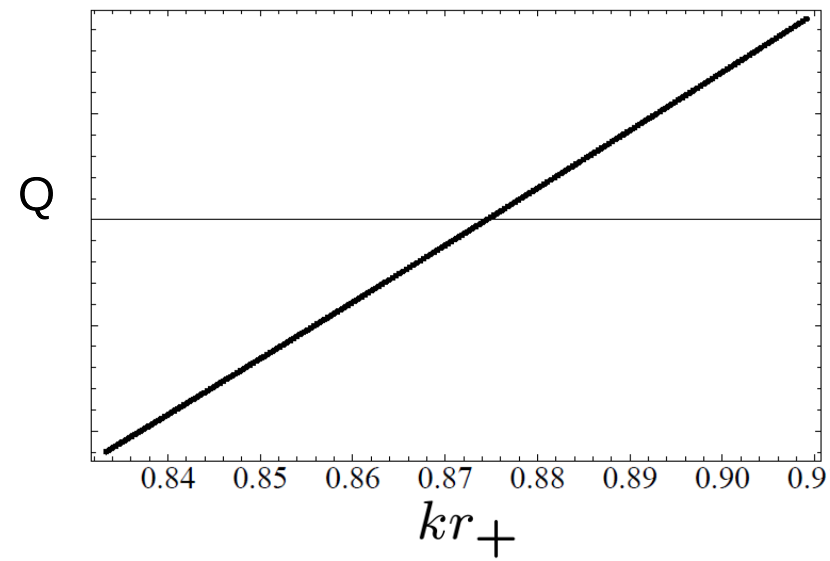
separation of the variable $\dot{\Psi} = \psi(r) \cos kx$

$$Q \equiv \ddot{M} - T\ddot{A}/4 - \frac{1}{T\,c_J}\dot{M}^2$$

Result of GL instability

solution





black string is unstable for

k r_+ < 0.8745 (ours)

k r_+ < 0.8762 (GL)

0.2% difference

(This is not numerical error.)

other dimensions

d	5	6	7	8	9	10	11
$(r_+/L)_{\rm crit}$	0.8745	1.2665	1.5779	1.8454	2.0837	2.3006	2.5007
$(r_+/L)_{\rm GL}$	0.8762	1.2689	1.5808	1.8486	2.0872	2.3041	2.5044

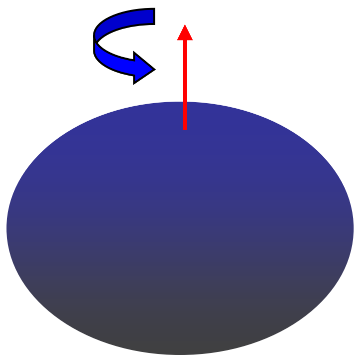
Although our technique gives only sufficient condition of instability, our result predicts GL instability with good accuracy.

4.Instability of Singly spinning Myers-Perry BHs

Singly spinning Myers-Perry black holes

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}(d\phi - \Omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\Omega^2_{(d-4)} \ ,$$

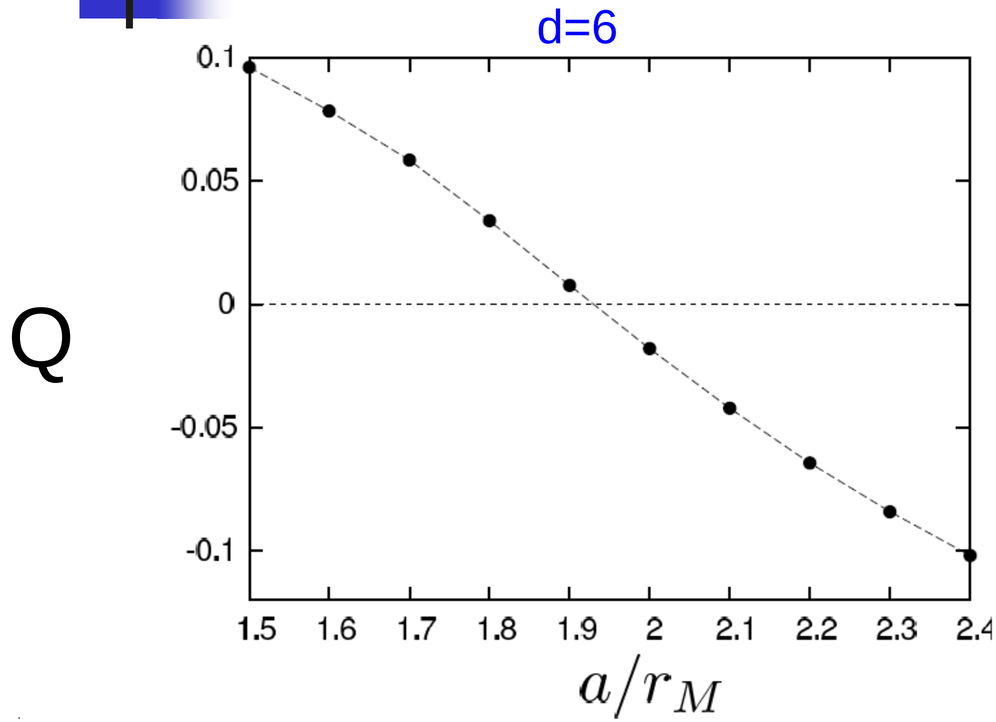
$$\left(\begin{array}{ll} \Delta = r^2 + a^2 - \dfrac{r_M^{d-3}}{r^{d-5}} \ , & \rho^2 = r^2 + a^2\cos^2\theta \ , \\ \Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta \ , & \Omega = \dfrac{r_M^{d-3}a}{\Sigma^2 r^{d-5}} \end{array}\right)$$



We consider axi-symmetric perturbations on this spacetime.

Stability of this mode has been already studied by Dias et al(2009).

Instability of Myers-Perry BHs



We found instability for

$$a/r_M > 1.933$$

On the other hand,
the result of Dias et al is

$$a/r_M > 1.572$$

Our result is consistent with Dias et al.

d	6	7	8	9	10	11
a/r_M	1.933	2.380	2.635	2.803	2.934	3.048
a/r_M(Dias et al.)	1.572	1.714	1.770	1.792	1.795	1.798

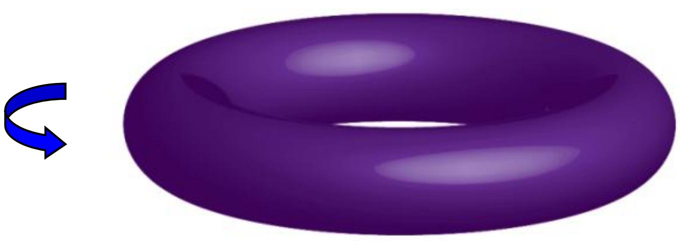
5. Instability of fat black rings

Singly spinning black rings

Emparan & Reall, 01

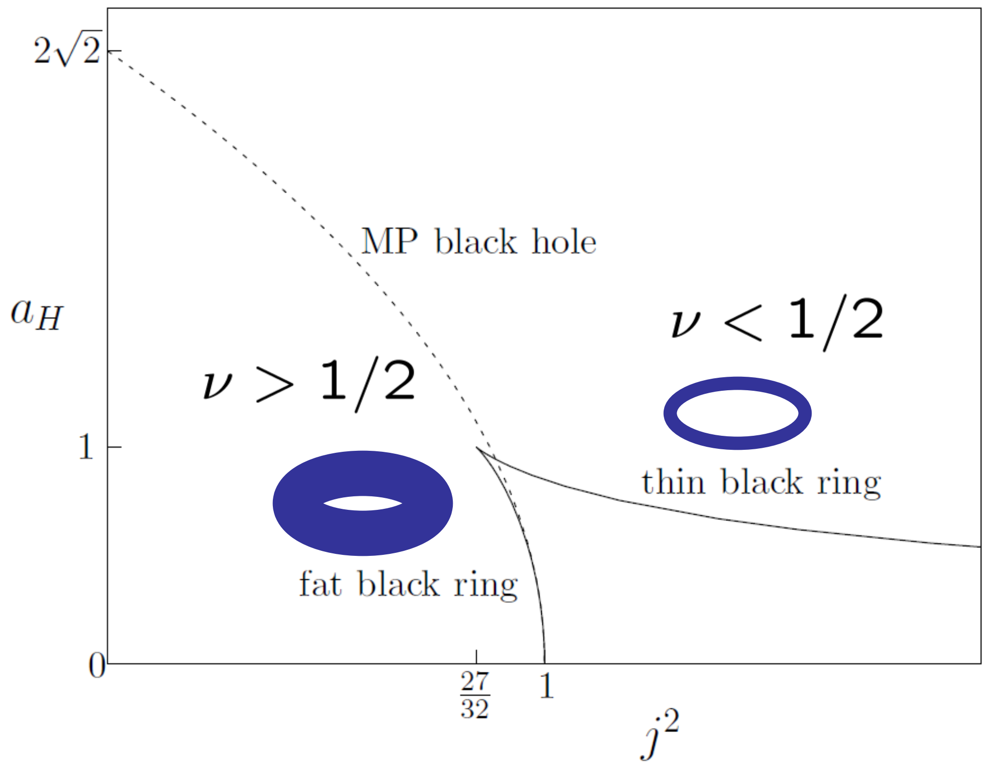
$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\phi_2\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\phi_2^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi_1^2\right]$$

$$\left(F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi), \qquad C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}} \right)$$

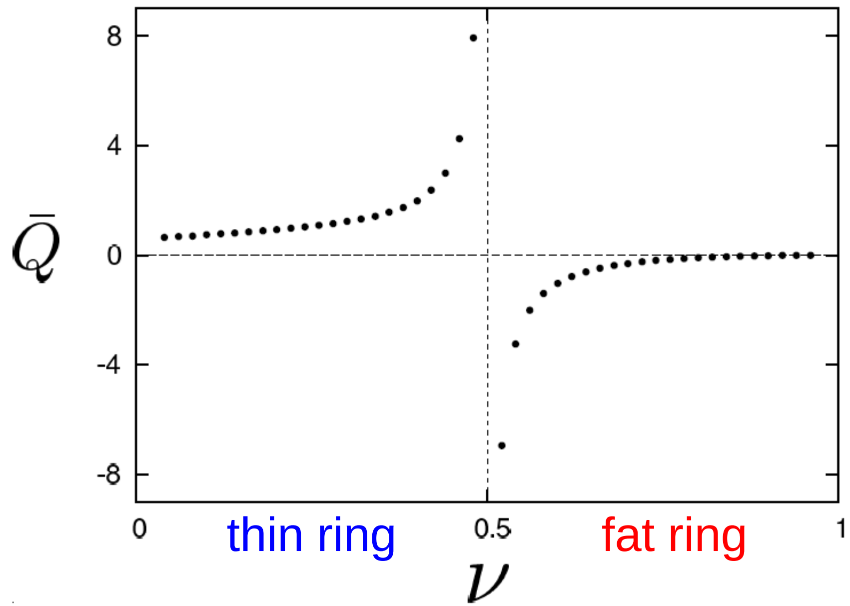


We consider axi-symmetric perturbations on this spacetime.

phase diagram of black rings



fat rings are unstable



Q<0 for v>1/2. Therefore, fat rings are unstable.

At nu=1/2, Q is diversing, because of $c_J = 0 \quad (\text{for } \nu = 1/2)$

$$Q \equiv \ddot{M} - T\ddot{A}/4 - \frac{1}{T\,c_J}\dot{M}^2 \qquad \left(c_J = \left(\frac{\partial M}{\partial T}\right)_J \right)$$

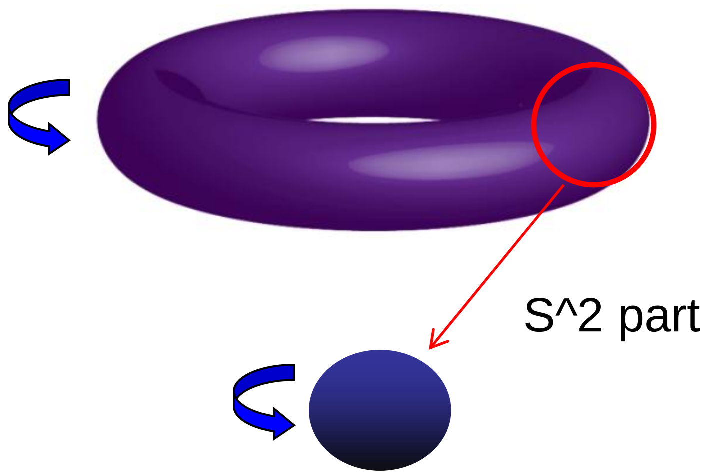
doubly spinning black rings

Pomeransky&Senkov, 06

$$ds^2 = -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\phi^2 - 2\frac{J(x,y)}{H(y,x)}d\phi d\psi + \frac{F(y,x)}{H(y,x)}d\psi^2 + \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right).$$

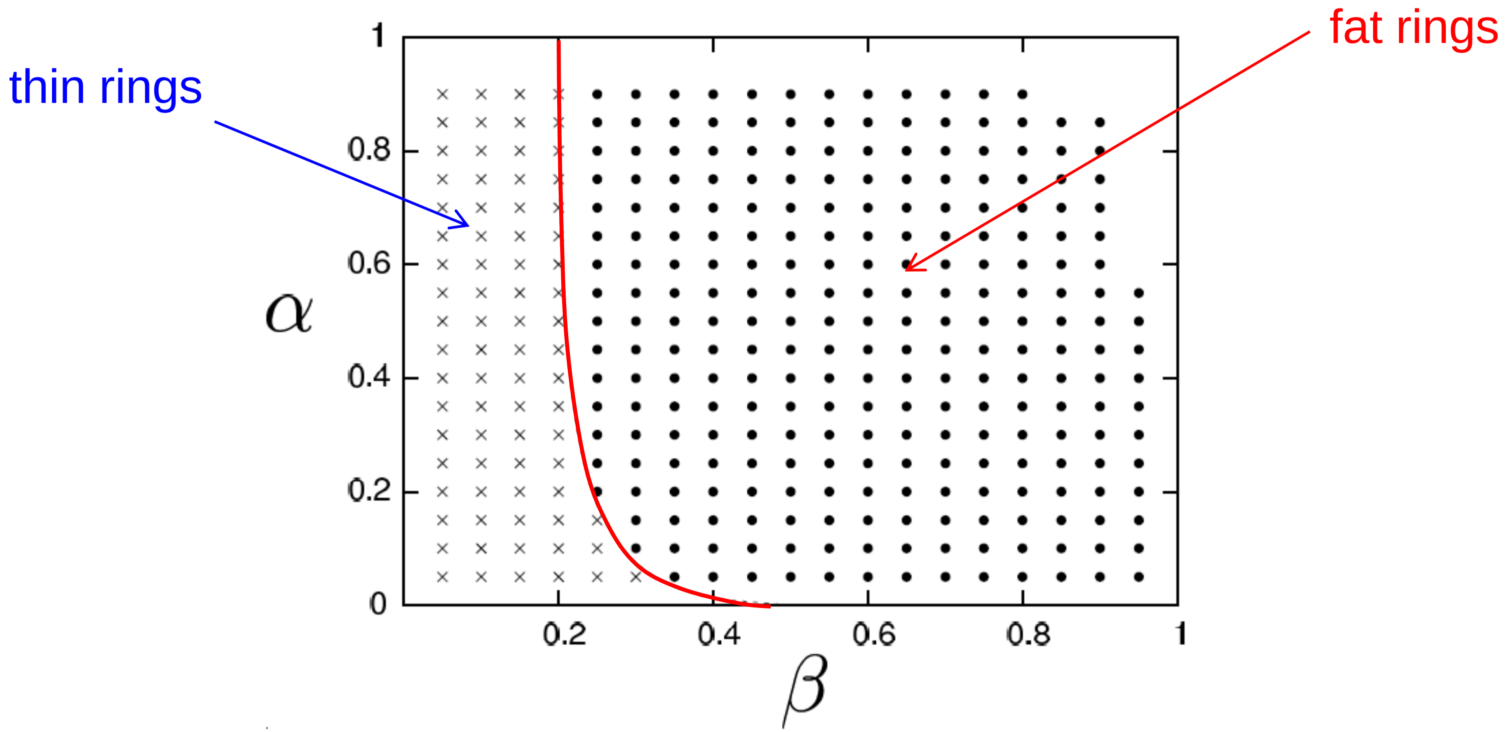
$$\Omega = -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)}\Big[(1-x^2)y\sqrt{\nu}d\psi + \frac{1+y}{1-\lambda+\nu}\{1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y)\}d\phi\Big],$$

$$G(x) = (1-x^2)(1+\lambda x+\nu x^2)\,,$$
$$H(x,y) = 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2x\lambda(1-y^2\nu^2)+x^2y^2\nu(1-\lambda^2-\nu^2)\,,$$
$$J(x,y) = \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}\{1+\lambda^2-\nu^2+2(x+y)\lambda\nu-xy\nu(1-\lambda^2-\nu^2)\},$$
$$F(x,y) = \frac{2k^2}{(x-y)^2(1-\nu)^2}\Big[G(x)(1-y^2)\{\left((1-\nu)^2-\lambda^2\right)(1+\nu)+y\lambda(1-\lambda^2+2\nu-3\nu^2)\} + G(y)\{2\lambda^2+x\lambda((1-\nu)^2+\lambda^2)+x^2((1-\nu)^2-\lambda^2)(1+\nu) + x^3\lambda(1-\lambda^2-3\nu^2+2\nu^3)-x^4(1-\nu)\nu(-1+\lambda^2+\nu^2)\}\Big]\,.$$



fat doubly spinning black rings are unstable

DSBRs are two parameter family. (α, β)



The points $\bullet$ and $\times$ correspond to $Q < 0$ and $Q > 0$, respectively.

6. Summary

- We study black hole instabilities in the view of local Penrose inequality.
- We reproduced Gregory-Laflamme instability with good accuracy.
- We showed instability of Singly spinning Myers-Perry BHs.
  (This supports result of Dias et al.)
- Singly and Doubly spinning black rings are unstable in fat branch.

Future work

- non-uniform black string

  Work in progress.
  We found a stable phase in non-uniform black strings in D=12, 13.
- AdS black holes
- holographic superconductor

contents

Why higher dimensional BHs?

- string theory → higher dimensional general relativity
  low energy limit
  - Higher dimensional BHs are solutions of a low energy effective theory of string theory.
  - Quantum aspects of higher dimensional BHs can be understood in the view of string theory.
- (Higher dimensional) black holes in AdS ⟺ A gauge theory with finite temperature
  - AdS BHs correpond to thermal states of dual theory.
  - In recent years, the AdS/CFT is applied to realistic systems such as QCD or condensed matter physics.
- black hole production at LHC.

Recent progress of stability analysis (all for Myers-Perry BHs)

numerical difficulty

dimension	angular momenta	perturbation equations	result	ref
any dim	$J_1=..=J_n=0$	ODE	stable	Ishibashi &Kodama,03
D=5	$J_1=J_2$	ODE	no evidence of instability	KM&Soda, 09
D=7,9,11,...	$J_1=..=J_n$	ODE	unstable for large J	Dias et al, 10
D>=6	$J_1\neq0$ $J_1=..=J_n=0$	PDE	unstable for large J (axisymmetric mode)	Dias et al, 09
D>=5	$J_1\neq0$ $J_1=..=J_n=0$	PDE	unstable for large J (non-axisymmetric mode)	Yoshino &Shibata,10

We need easier method to study the black hole stability.

Recent progress of stability analysis (all for Myers-Perry BHs)

numerical difficulty

- any dimensions $J_1 = J_2 = \cdots = J_n = 0$

  Schwarzshild BHs are stable.(Ishibashi & Kodama, 03)
- D=5 $J_1 = J_2$

  There is no evidence of instability. (KM & Soda, 08)
- D=7, 9,11, ... $J_1 = J_2 = \cdots = J_n$

  Unstable for large J. (Dias, Figueras, Monteiro, Reall & Santos, 09)

Perturbation equations are separable

ODE

- D >= 6 $J_1 \neq 0, \quad J_2 = \cdots = J_n = 0$

  Unstable against axisymmetric perturbations.

  (Dias, Figueras, Monteiro, Santos & Emparan, 09)
- D >= 5 $J_1 \neq 0, \quad J_2 = \cdots = J_n = 0$

  Unstable against non-axisymmetric perturbations.

  (Yoshino and Shibata, 10)

Perturbation equations are not separable

PDE

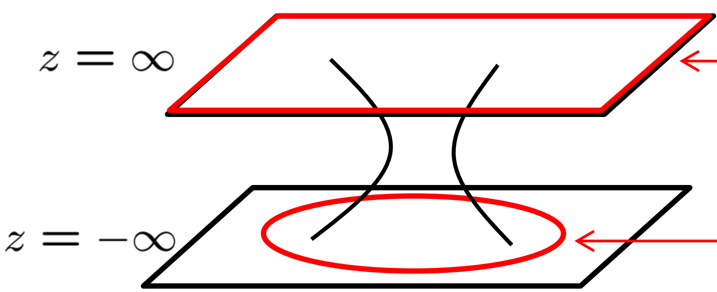
We need an easier method to study the black hole stability.

Boundary conditions

metric on t = const

$$ds^2 = \frac{\Sigma^2 \sin^2\theta}{\rho^2} d\phi^2 + \underline{\frac{4r_+^2 z^2 \rho^2}{\Delta}} dz^2 + \rho^2 d\theta^2 + r^2 \cos^2\theta \, d\Omega_{(d-4)}^2 \qquad \left(z^2 = \frac{r - r_+}{r_+} \right)$$

In z-coordinate, the horizon is regular.



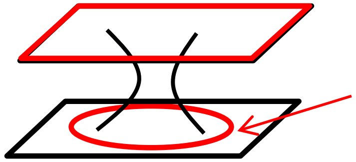
$$\boxed{\dot{\Psi}, \ \ddot{\Psi} \sim r^{-d+3}}$$

$$\boxed{\dot{\Psi} = 1 + \sum_{n=1}^{N} c_n \cos 2n\theta \qquad \ddot{\Psi} = 0}$$

At inner boundary, we can chose any boundary condition.
Inner boundary condition is parameterized by Fourier coefficients.
These c_n's are determined later.

Instability of Myers-Perry BHs

Inner boundary condition was parameterized by Fourier coefficients.

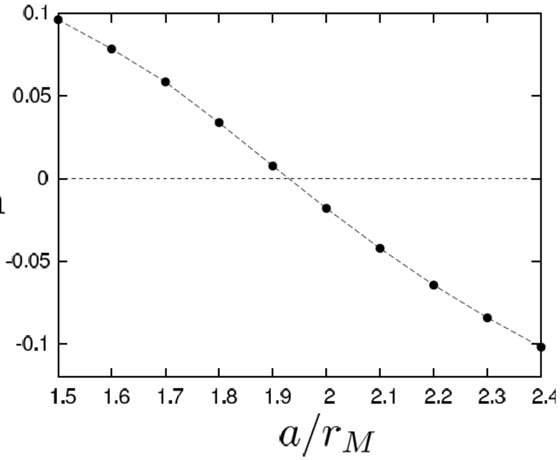


$$\dot{\Psi} = 1 + \sum_{n=1}^{N} c_n \cos 2n\theta$$

The Q can be regarded as a function of these coefficients. $Q = Q(c_1, c_2, \cdots)$

⇒ minimization of Q

minimum value of Q for d=6



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For 2nd order eq $\bar{\nabla}^2\ddot{\Psi} = 0$,

we can chose the trivial solution $\ddot{\Psi} = 0$,

because homogeneous parts of $\ddot{\Psi}$ does not affect Q.